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THE STEIN-WEISS TYPE INEQUALITY FOR FRACTIONAL INTEGRALS, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR *)

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Abstract

In this paper we study the Riesz potentials (B-Riesz potentials) generated by the Laplace-Bessel differential operator $\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$, $\gamma > 0$, in the weighted Lebesgue spaces $L_{p,|x|^\beta,\gamma}$. We establish an inequality of Stein-Weiss type for the B-Riesz potentials, and obtain necessary and sufficient conditions on the parameters for the boundedness of the B-Riesz potential operator from the spaces $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^\lambda,\gamma}$, and from the spaces $L_{1,|x|^\beta,\gamma}$ to the weak spaces $WL_{q,|x|^\lambda,\gamma}$.

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Key Words and Phrases: Laplace-Bessel differential operator, generalized shift operator, B-Riesz potential, Stein-Weiss type inequality, weighted Lebesque space

0. Introduction

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The potential and related topics associated with the Laplace-Bessel differential operator

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$$\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0$$

have been the research areas many mathematicians such as B. Muckenhoupt and E. Stein [7], I. Kipriyanov [8], L. Lyakhov [10], K. Stempak [12], A.D. Gadjiev and I.A. Aliev [1], V.S. Guliyev [2]-[4], and others.

In this paper we study Riesz potentials (B-Riesz potentials) generated by the Laplace-Bessel differential operator Δ_B in weighted Lebesque spaces. We establish an inequality of Stein-Weiss type (see [11]) for the B-Riesz potentials. We obtain the necessary and sufficient conditions on the parameters for boundedness of the B-Riesz potential operator from the spaces $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^\lambda,\gamma}$, and from the spaces $L_{1,|x|^\beta,\gamma}$ to the weak spaces $WL_{q,|x|^\lambda,\gamma}$.

1. Definitions, notation and preliminaries

Let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n ; \ x = (x_1, ..., x_n), x_n > 0\}$ and $B(x,r) = \{y \in \mathbb{R}_+^n \ : \ |x-y| < r, \ r > 0\}$, $B_r \equiv B(0,r)$, and let ${}^{\complement}B(x,r) = \mathbb{R}_+^n \setminus B(x,r)$.

For a measurable set $A \subset \mathbb{R}_+^n$ let $|A|_\gamma = \int_A x_n^\gamma dx$, then $|B_r|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where

$$\omega(n,\gamma) = \int_{B_1} x_n^\gamma dx = \frac{\pi^{(n-1)/2}\Gamma\left((\gamma+1)/2\right)}{2\Gamma\left((n+\gamma-2)/2\right)}.$$

Denote by T^y the generalized shift operator (B–shift operator) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x'-y', (x_n, y_n)_\beta\right) \sin^{\gamma-1}\beta d\beta,$$

where $(x_n,y_n)_\beta = \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\beta}$ and $C_\gamma = \frac{\Gamma((\gamma+1)/2)}{\sqrt{\pi}\Gamma(\gamma/2)} = \frac{2}{\pi}\,\omega(2,\gamma)$.

We note that the generalized shift operator T^y is closely connected with the Laplace-Bessel differential operator Δ_B (for example, $n=1$ see [9] and $n>1$ [8] for details).

Let $L_{p,\gamma}(\mathbb{R}_+^n)$ be the space of measurable functions on $\mathbb{R}_+^n$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}_+^n)} = \left(\int_{\mathbb{R}_+^n} |f(x)|^p x_n^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

For $p=\infty$ the space $L_{\infty,\gamma}(\mathbb{R}_+^n)$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^n} |f(x)|.$$

The translation operator T^y generates the corresponding B -convolution
$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y)\, T^y g(x)\, y_n^\gamma dy,$$
for which the Young inequality holds:
$$\|f \otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p, q, r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

LEMMA 1. *Let* $0 < \alpha < n + \gamma$. *Then*
$$\left|T^y |x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| \le 2^{n+\gamma+1-\alpha} |y|^{\alpha-n-\gamma-1} |x| \tag{1}$$
for $2|x| \le |y|$.

P r o o f. We will show that
$$\begin{aligned} &\left|T^y |x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| \\ \le C_\gamma \int_0^\pi &\left| \left|(x' - y', (x_n, y_n)_\beta)\right|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| \sin^{\gamma-1}\beta d\beta. \end{aligned}$$

From the mean value theorem we have
$$\begin{aligned} &\left| \left|(x' - y', (x_n, y_n)_\beta)\right|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma}\right| \\ &\le \left| \left|(x' - y', (x_n, y_n)_\beta)\right| - |y|\right| \xi^{\alpha-n-\gamma-1}, \end{aligned}$$
where $\min\{|(x' - y', (x_n, y_n)_\beta)|, |y|\} \le \xi \le \max\{|(x' - y', (x_n, y_n)_\beta)|, |y|\}$.
Note that
$$\left|(x' - y', (x_n, y_n)_\beta)\right| \le |x| + |y| \le \frac{3}{2}|y|,$$
$$\left|(x' - y', (x_n, y_n)_\beta)\right| \ge |x - y| \ge |y| - |x| \ge \frac{1}{2}|y|$$
and
$$\begin{aligned} \left|(x' - y', (x_n, y_n)_\beta)\right| - |y| &\le |x| + |y| - |y| \le |x| \\ |y| - \left|(x' - y', (x_n, y_n)_\beta)\right| &\le |y| - |x - y| \le |x|. \end{aligned}$$
Hence
$$\frac{1}{2}|y| \le \left|(x' - y', (x_n, y_n)_\beta)\right| \le \frac{3}{2}|y|, \text{ and } \left|\left|(x' - y', (x_n, y_n)_\beta)\right| - |y|\right| \le |x|.$$

■

Definition 1. Let $1 \le p < \infty$. We denote by $WL_{p,\gamma}(\mathbb{R}^n_+)$ the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions f with the finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r f_{*,\gamma}^{1/p}(r),$$

where $f_{*,\gamma}(r) = \left|\left\{x \in \mathbb{R}^n_+ \ : \ |f(x)| > r\right\}\right|_\gamma$.

Let v be non-negative and measurable function on $\mathbb{R}^n_+$, and $L_{p,v,\gamma}(\mathbb{R}^n_+)$ be the weighted $L_{p,\gamma}$-space of all measurable functions f on $\mathbb{R}^n_+$ for which

$$\|f\|_{L_{p,v,\gamma}} \equiv \|f\|_{L_{p,v,\gamma}(\mathbb{R}^n_+)} = \|vf\|_{L_{p,\gamma}(\mathbb{R}^n_+)} < \infty.$$

We denote by $WL_{p,v,\gamma}(\mathbb{R}^n_+)$ ($1 \le p < \infty$) the weighted weak Lebesgue space which is the class of all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$, for which

$$\|f\|_{WL_{p,v,\gamma}} \equiv \|f\|_{WL_{p,v,\gamma}(\mathbb{R}^n_+)} = \|vf\|_{WL_{p,\gamma}(\mathbb{R}^n_+)} < \infty.$$

We shall need the following Hardy-type transforms defined on $\mathbb{R}^n_+$:

$$H_\gamma f(x) = \int_{B_{|x|}} f(y) y_n^\gamma dy \quad , \quad H'_\gamma f(x) = \int_{\complement B_{|x|}} f(y) y_n^\gamma dy.$$

The following two theorems for these transformations were proved in [5] (see also [6], Section 1.1).

Theorem A. *Let* $1 < q < \infty$. *Suppose that* v *and* w *are a.e. positive functions on* $\mathbb{R}^n_+$. *Then:*

(a) *The operator* H_γ *is bounded from* $L_{1,w,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only if*

$$A_1 \equiv \sup_{t>0} \left(\int_{\complement B_t} v^q(x) x_n^\gamma dx \right)^{1/q} \sup_{B_t} w^{-1}(x) < \infty,$$

(b) *The operator* H'_γ *is bounded from* $L_{1,w,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only if*

$$A_2 \equiv \sup_{t>0} \left(\int_{B_t} v^q(x) x_n^\gamma dx \right)^{1/q} \sup_{\complement B_t} w^{-1}(x) < \infty.$$

Moreover, there exist positive constants a_j, $j = 1, \ldots, 4$, *depending only on* q *such that* $a_1 A_1 \le \|H\| \le a_2 A_1$ *and* $a_3 A_2 \le \|H'\| \le a_4 A_2$.

Theorem B. *Let* $1 < p \le q < \infty$. *Suppose that* v *and* w *are a.e. positive functions on* $\mathbb{R}^n_+$. *Then:*

(a) *The operator* H_γ *is bounded from* $L_{p,w,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,v,\gamma}(\mathbb{R}^n_+)$ *if and only if*

$$A_3 \equiv \sup_{t>0} \left(\int_{\complement B_t} v^q(x) x_n^\gamma dx \right)^{1/q} \left(\int_{B_t} w^{-p'}(x) x_n^\gamma dx \right)^{1/p'} < \infty,\ p' = p/(p-1),$$

(b) The operator H'_γ is bounded from $L_{p,w,\gamma}(\mathbb{R}^n_+)$ to $L_{q,v,\gamma}(\mathbb{R}^n_+)$ if and only if

$$A_4 \equiv \sup_{t>0} \left(\int_{B_t} v^q(x) x_n^\gamma dx \right)^{1/q} \left(\int_{\complement B_t} w^{-p'}(x) x_n^\gamma dx \right)^{1/p'} < \infty.$$

Moreover, there exist positive constants b_j , $j = 1, \dots, 4$, depending only on p and q such that $b_1 A_3 \le \|H\| \le b_2 A_3$ and $b_3 A_4 \le \|H'\| \le b_4 A_4$.

We will need the case when we substitute $L_{p,v,\gamma}(\mathbb{R}^n_+)$ by the homogeneous space (X, ρ, μ) , $X = \mathbb{R}^n_+$, $\rho(x-y) = |x-y|$, $d\mu(x) = x_n^\gamma dx$ in Theorems A and B.

Consider the B -Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_+} T^y |x|^{\alpha-n-\gamma} f(y) y_n^\gamma dy, \quad 0 < \alpha < n + \gamma.$$

For the B -Riesz potential the following Hardy-Littlewood-Sobolev theorem is valid.

Theorem 1. ([1]) *Let $0 < \alpha < n+\gamma$ and $1 \le p < \frac{n+\gamma}{\alpha}$.*

1) If $1 < p < \frac{n+\gamma}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $L_{q,\gamma}(\mathbb{R}^n_+)$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n+\gamma}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{1,\gamma}(\mathbb{R}^n_+)$ to $WL_{q,\gamma}(\mathbb{R}^n_+)$.

2. Main results

One of the our main results is the following Stein-Weiss type theorem for the B -Riesz potentials.

Theorem 2. *Let $0 < \alpha < n+\gamma$, $1 < p \le q < \infty$, $\beta < \frac{n+\gamma}{p'}$, $\lambda < \frac{n+\gamma}{q}$, $\beta + \lambda \ge 0$ ($\beta + \lambda > 0$, if $p = q$), $\frac{1}{p} - \frac{1}{q} = \frac{\alpha-\beta-\lambda}{n+\gamma}$ and $f \in L_{p,|x|^\beta,\gamma}(\mathbb{R}^n_+)$. Then $I_{\alpha,\gamma} f \in L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$ and the following inequality holds:*

$$\left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left| I_{\alpha,\gamma} f(x) \right|^q x_n^\gamma dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n_+} |x|^{\beta p} |f(x)|^p x_n^\gamma dx \right)^{1/p}, \quad (2)$$

where C is independent of f .

P r o o f. We have

$$\left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left|I_{\alpha,\gamma} f(x)\right|^q x_n^\gamma dx\right)^{1/q}$$
$$\leq \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(\int_{B_{|x|/2}} |f(y)|\, T^y |x|^{\alpha-n-\gamma} y_n^\gamma dy\right)^q x_n^\gamma dx\right)^{1/q}$$
$$+ \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|\, T^y |x|^{\alpha-n-\gamma} y_n^\gamma dy\right)^q x_n^\gamma dx\right)^{1/q}$$
$$+ \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(\int_{\complement B_{2|x|}} |f(y)|\, T^y |x|^{\alpha-n-\gamma} y_n^\gamma dy\right)^q x_n^\gamma dx\right)^{1/q} \equiv I_1 + I_2 + I_3.$$

It is easy to verify that if $|y| < |x|/2$, then $|x| \leq |y| + |x-y| < |x|/2 + |x-y|$. Hence $|x|/2 < |x-y|$ and $T^y|x|^{\alpha-n-\gamma} \leq (|x|/2)^{\alpha-n-\gamma}$. Consequently,

$$I_1 \leq 2^{n+\gamma-\alpha} \left(\int_{\mathbb{R}^n_+} |x|^{(\alpha-n-\gamma-\lambda)q} \left(H_\gamma f(x)\right)^q x_n^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q < (n+\gamma-\alpha)q - n - \gamma$ ($\equiv \alpha < \frac{n+\gamma}{q'} + \lambda$) we have

$$\left(\int_{\complement B_t} |x|^{(-\lambda+\alpha-n-\gamma)q} x_n^\gamma dx\right)^{1/q} = C_1 t^{\alpha-\lambda-(n+\gamma)/q'},$$

where $C_1 = \left(\frac{\omega(n,\gamma)}{q/q'+(\lambda-\alpha)q/(n+\gamma)}\right)^{1/q}$.

Analogously, by virtue of the condition $\beta p < (n+\gamma)(p-1)$ ($\equiv \beta < \frac{n+\gamma}{p'}$), it follows that

$$\left(\int_{B_t} |x|^{-\beta p'} x_n^\gamma dx\right)^{1/p'} = C_2 t^{(n+\gamma)/p'-\beta},$$

where $C_2 = \left(\frac{\omega(n,\gamma)}{1-\beta p'/(n+\gamma)}\right)^{1/p'}$.

Summarizing these estimates, we find that

$$\sup_{t>0} \left(\int_{\complement B_t} |x|^{(-\lambda+\alpha-n-\gamma)q} x_n^\gamma dx\right)^{1/q} \left(\int_{B_t} |x|^{-\beta p'} x_n^\gamma dx\right)^{1/p'}$$
$$= C_1 C_2 \sup_{t>0} t^{\alpha-\beta-\lambda+\frac{n+\gamma}{q}-\frac{n+\gamma}{p}} < \infty \Longleftrightarrow \alpha - \beta - \lambda = \frac{n+\gamma}{p} - \frac{n+\gamma}{q}.$$

Now the first part of Theorem A leads us to the inequality

$$I_1 \le b_2 C_1 C_2 2^{n+\gamma-\alpha} \left(\int_{\mathbb{R}^n_+} |x|^\beta |f(x)|^p x_n^\gamma dx \right)^{1/p}.$$

It is easy to verify that if $|y| > 2|x|$, then $|y| \le |x| + |x-y| < |y|/2 + |x-y|$. Hence $|y|/2 < |x-y|$ and $T^y |x|^{\alpha-n-\gamma} \le (|y|/2)^{\alpha-n-\gamma}$. Consequently,

$$I_3 \le 2^{n+\gamma-\alpha} \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(H'_\gamma \left(|f(y)||y|^{\alpha-n-\gamma} \right)(x) \right)^q x_n^\gamma dx \right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q > -n-\gamma$ ($\equiv \lambda < \frac{n+\gamma}{q}$), we have

$$\left(\int_{B_t} |x|^{-\lambda q} x_n^\gamma dx \right)^{1/q} = C_3 t^{(n+\gamma)/q-\lambda},$$

where $C_3 = \left(\frac{\omega(n,\gamma)}{1-\lambda q/(n+\gamma)} \right)^{1/q}$. Analogously, by virtue of the condition $\beta p > \alpha p - n - \gamma$ ($\equiv \alpha < \frac{n+\gamma}{p} + \beta$), it follows that

$$\left(\int_{B_t} |x|^{-(\beta+n+\gamma-\alpha)p'} x_n^\gamma dx \right)^{1/p'} = C_4 t^{(n+\gamma)/p' - (n+\gamma+\beta-\alpha)},$$

where $C_4 = \left(\frac{\omega(n,\gamma)}{(1+(\beta-\alpha)/(n+\gamma))p'-1} \right)^{1/p'}$.

Summarizing these estimates, we find that

$$\sup_{t>0} \left(\int_{B_t} |x|^{-\lambda q} x_n^\gamma dx \right)^{1/q} \left(\int_{\complement B_t} |x|^{-(\beta+n+\gamma-\alpha)p'} x_n^\gamma dx \right)^{1/p'}$$
$$= C_3 C_4 \sup_{t>0} t^{\alpha-\beta-\lambda+\frac{n+\gamma}{q}-\frac{n+\gamma}{p}} < \infty \Longleftrightarrow \alpha - \beta - \lambda = \frac{n+\gamma}{p} - \frac{n+\gamma}{q}.$$

Now the second part of Theorem B leads us to the inequality

$$I_3 \le b_4 C_3 C_4 2^{n+\gamma-\alpha} \left(\int_{\mathbb{R}^n_+} |x|^\beta |f(x)|^p x_n^\gamma dx \right)^{1/p}.$$

To estimate I_2 , we consider the cases $\alpha < \frac{n+\gamma}{p}$ and $\alpha > \frac{n+\gamma}{p}$ separately.

Let $\alpha < \frac{n+\gamma}{p}$. In this case the condition $\alpha = \beta + \lambda + \frac{n+\gamma}{p} - \frac{n+\gamma}{q} \ge \frac{n+\gamma}{p} - \frac{n+\gamma}{q}$ implies $q \le p^*$, where $p^* = (n+\gamma)p/(n+\gamma-\alpha p)$.

First, assume that $q < p^*$. In the sequel we use the notation

$$D_k \equiv \{x \in \mathbb{R}^n_+ : 2^k \le |x| < 2^{k+1}\} \ \ , \ \ \widetilde{D_k} \equiv \{x \in \mathbb{R}^n_+ : 2^{k-2} \le |x| < 2^{k+2}\}.$$

By Hölder's inequality with respect to the exponent $\frac{p^*}{q}$ and Theorem 1, we find that

$$I_2 = \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(\int_{B_{2|x|} \setminus B_{|x|/2}} |f(y)| \, T^y |x|^{\alpha - n - \gamma} y_n^\gamma dy \right)^q x_n^\gamma dx \right)^{1/q}$$

$$= \left(\sum_{k \in \mathbb{Z}} \int_{D_k} |x|^{-\lambda q} \left(\int_{B_{2|x|} \setminus B_{|x|/2}} |f(y)| \, T^y |x|^{\alpha - n - \gamma} y_n^\gamma dy \right)^q x_n^\gamma dx \right)^{1/q}$$

$$\le \left(\sum_{k \in \mathbb{Z}} \left(\int_{D_k} \left(\int_{B_{2|x|} \setminus B_{|x|/2}} |f(y)| \, T^y |x|^{\alpha - n - \gamma} y_n^\gamma dy \right)^{p^*} x_n^\gamma dx \right)^{q/p^*} \right.$$

$$\left. \times \left(\int_{D_k} |x|^{\frac{-\lambda q p^*}{p^* - q}} x_n^\gamma dx \right)^{\frac{p^* - q}{p^*}} \right)^{1/q}$$

$$\le C_5 \left(\sum_{k \in \mathbb{Z}} 2^{k[-\lambda q + \frac{p^* - q}{p^*}(n + \gamma)]} \left(\int_{D_k} \left| I_{\alpha,\gamma} \left(f \chi_{\widetilde{D_k}} \right)(x) \right|^{p^*} x_n^\gamma dx \right)^{q/p^*} \right)^{1/q}$$

$$\le C_6 \left(\sum_{k \in \mathbb{Z}} 2^{k[-\lambda q + \frac{p^* - q}{p^*}(n + \gamma)]} \left(\int_{\widetilde{D_k}} |f(x)|^p x_n^\gamma dx \right)^{q/p} \right)^{1/q}$$

$$\le C_7 \left(\int_{\mathbb{R}^n_+} |x|^\beta |f(x)|^p x_n^\gamma dx \right)^{1/p} .$$

If $q = p^*$, then $\beta + \lambda = 0$ and consequently, using directly Theorem 1 we have

$$I_2 \le C_8 \left(\sum_{k \in \mathbb{Z}} 2^{k \beta p^*} \int_{D_k} \left| I_{\alpha,\gamma} \left(f \chi_{\widetilde{D_k}} \right)(x) \right|^{p^*} x_n^\gamma dx \right)^{1/p^*}$$

$$\le C_9 \left(\sum_{k \in \mathbb{Z}} 2^{k \beta p^*} \left(\int_{\widetilde{D_k}} |f(x)|^p x_n^\gamma dx \right)^{p^*/p} \right)^{1/p^*}$$

$$\le C_{10} \left(\int_{\mathbb{R}^n_+} |x|^{\beta p} |f(x)|^p x_n^\gamma dx \right)^{1/p} .$$

Now let $\alpha > \frac{n+\gamma}{p}$. In this case by Hölder's inequality with respect to the exponent p we get the following estimate

$$I_2 \le \left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} \left(\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|^p y_n^\gamma dy \right)^{q/p} \times \left(\int_{B_{2|x|}\setminus B_{|x|/2}} \left(T^y |x|^{\alpha-n-\gamma} \right)^{p'} y_n^\gamma dy \right)^{q/p'} x_n^\gamma dx \right)^{1/q} .$$

On the other hand, using (2) and the inequality $\alpha > \frac{n+\gamma}{p}$, we find that

$$\int_{B_{2|x|}\setminus B_{|x|/2}} \left(T^y |x|^{\alpha-n-\gamma} \right)^{p'} y_n^\gamma dy \le \int_{B_{2|x|}\setminus B_{|x|/2}} |x-y|^{(\alpha-n-\gamma)p'} y_n^\gamma dy$$

$$\le \int_0^\infty \left| B_{2|x|} \cap E(x, \tau^{\frac{1}{(\alpha-n-\gamma)p'}}) \right|_\gamma d\tau$$

$$\le \int_0^{|x|^{(\alpha-n-\gamma)p'}} \left| B_{2|x|} \right|_\gamma d\tau + \int_{|x|^{(\alpha-n-\gamma)p'}}^\infty \left| E(x, \tau^{\frac{1}{(\alpha-n-\gamma)p'}}) \right|_\gamma d\tau$$

$$\le C_{11} |x|^{(\alpha-n-\gamma)p'+n+\gamma} + C_{12} \int_{|x|^{(\alpha-n-\gamma)p'}}^\infty \tau^{\frac{1}{(\alpha-n-\gamma)p'}} d\tau = C_{13} |x|^{(\alpha-n-\gamma)p'+n+\gamma},$$

where the positive constant C_{13} does not depend on x . The latter estimate yields

$$I_2 \le C_{14} \left(\sum_{k\in\mathbb{Z}} \int_{D_k} |x|^{-\lambda q + [(\alpha-n-\gamma)p'+n+\gamma]\frac{q}{p'}} \left(\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|^p y_n^\gamma dy \right)^{q/p} x_n^\gamma dx \right)^{1/q}$$

$$\le C_{14} \left(\sum_{k\in\mathbb{Z}} \int_{D_k} \left(\int_{\widetilde{D_k}} |f(y)|^p y_n^\gamma dy \right)^{q/p} |x|^{-\lambda q + [(\alpha-n-\gamma)p'+n+\gamma]\frac{q}{p'}} x_n^\gamma dx \right)^{1/q}$$

$$\le C_{14} \left(\sum_{k\in\mathbb{Z}} 2^{k(-\lambda+\alpha-n-\gamma+\frac{n+\gamma}{p'}+\frac{n+\gamma}{q})q} \left(\int_{\widetilde{D_k}} |f(y)|^p y_n^\gamma dy \right)^{q/p} \right)^{1/q}$$

$$\le C_{14} \left(\sum_{k\in\mathbb{Z}} 2^{k\beta q} \left(\int_{\widetilde{D_k}} |f(x)|^p x_n^\gamma dx \right)^{q/p} \right)^{1/q} \le C_{15} \left(\int_{\mathbb{R}^n_+} |x|^{\beta p} |f(x)|^p x_n^\gamma dx \right)^{q/p} .$$

Thus Theorem 2 is completely proved. ■

To obtain the general result on the boundedness of the B -potentials $I_{\alpha,\gamma}$ we need the following weak weighted estimate.

THEOREM 3. *Let* $0 < \alpha < n+\gamma$, $1 < q < \infty$, $\beta \le 0$, $\lambda < \frac{n+\gamma}{q}$, $\beta + \lambda \ge 0$ $1 - \frac{1}{q} = \frac{\alpha-\beta-\lambda}{n+\gamma}$ *and* $f \in L_{1,|x|^\beta,\gamma}(\mathbb{R}^n_+)$. *Then* $I_{\alpha,\gamma} f \in WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$ *and the following inequality holds*

$$\left(\int_{\{x\in\mathbb{R}^n_+ :\, |x|^{-\lambda}|I_{\alpha,\gamma}f(x)|>\tau\}} x_n^\gamma dx\right)^{1/q} \le \frac{C}{\tau} \int_{\mathbb{R}^n_+} |x|^\beta |f(x)| x_n^\gamma dx, \qquad (3)$$

where C *is independent of* f .

P r o o f. We have

$$\left(\int_{\{x\in\mathbb{R}^n_+ :\, |x|^{-\lambda}|I_{\alpha,\gamma}f(x)|>\tau\}} x_n^\gamma dx\right)^{1/q}$$

$$\le \left(\int_{\{x\in\mathbb{R}^n_+ :\, |x|^{-\lambda}\int_{B_{|x|/2}} |f(y)|\, T^y|x|^{\alpha-n-\gamma} y_n^\gamma dy>\tau/3\}} x_n^\gamma dx\right)^{1/q}$$

$$+ \left(\int_{\{x\in\mathbb{R}^n_+ :\, |x|^{-\lambda}\int_{B_{2|x|}\setminus B_{|x|/2}} |f(y)|\, T^y|x|^{\alpha-n-\gamma} y_n^\gamma dy>\tau/3\}} x_n^\gamma dx\right)^{1/q}$$

$$+ \left(\int_{\{x\in\mathbb{R}^n_+ :\, |x|^{-\lambda}\int_{\complement B_{2|x|}} |f(y)|\, T^y|x|^{\alpha-n-\gamma} y_n^\gamma dy>\tau/3\}} x_n^\gamma dx\right)^{1/q} \equiv J_1 + J_2 + J_3.$$

Then

$$J_1 \le \left(\int_{\{x\in\mathbb{R}^n_+ :\, 2^{n+\gamma-\alpha}|x|^{\alpha-n-\gamma-\lambda} H_\gamma f(x)>\tau/3\}} x_n^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q < (n+\gamma-\alpha)q - n - \gamma$ ($\equiv \alpha < n+\gamma - \frac{n+\gamma}{q} + \lambda$) we have

$$\int_{\complement B_t} |x|^{(-\lambda+\alpha-n-\gamma)q} x_n^\gamma dx$$

$$= \int_{S^{n-1}_+} \int_t^\infty r^{(-\lambda+\alpha-n-\gamma)q+n+\gamma-1} \xi_n^\gamma d\xi dr = C_{16}\, t^{(-\lambda+\alpha-n-\gamma)q+n+\gamma},$$

where the positive constant C_{16} depends only on α , λ and q . Analogously by virtue of the condition $\beta \le 0$ it follows that

$$\sup_{B_t} |x|^{-\beta} = t^{-\beta}.$$

Summarizing these estimates we find that

$$\sup_{t>0}\left(\int_{\complement B_t}|x|^{(-\lambda+\alpha-n-\gamma)q}x_n^\gamma dx\right)^{1/q}\sup_{B_t}|x|^{-\beta}$$
$$=C_{16}\sup_{t>0}t^{\frac{n+\gamma}{q}-\lambda+\alpha-n-\gamma-\beta}<\infty\Longleftrightarrow\alpha-\beta-\lambda=n+\gamma-\frac{n+\gamma}{q}.$$

Now in the case $p=1$ the first part of Theorem A leads us to the inequality

$$J_1\leq\frac{C_{17}}{\tau}\int_{\mathbb{R}^n_+}|x|^\beta|f(x)|^p x_n^\gamma dx,$$

where the positive constant C_{17} is independent of f.

Also,

$$J_3\leq\left(\int_{\{x\in\mathbb{R}^n_+:\,2^{n+\gamma-\alpha}|x|^{-\lambda}H'_\gamma(|f(y)||y|^{\alpha-n-\gamma})(x)>\tau/3\}}x_n^\gamma dx\right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q>-n-\gamma$ ($\equiv\lambda<\frac{n+\gamma}{q}$) we have

$$\int_{B_t}|x|^{-\lambda q}x_n^\gamma dx=\int_{S_+^{n-1}}\int_0^t r^{-\lambda q+n+\gamma-1}\xi_n^\gamma d\xi dr=C_{18}t^{-\lambda q+n+\gamma},$$

where the positive constant C_{18} depends only on α and λ. Analogously by virtue of the condition $\beta\geq\alpha-n-\gamma$ it follows that

$$\sup_{\complement B_t}|x|^{-\beta+\alpha-n-\gamma}=t^{-\beta+\alpha-n-\gamma}.$$

Summarizing these estimates, we find that

$$\sup_{t>0}\left(\int_{B_t}|x|^{-\lambda q}x_n^\gamma dx\right)^{1/q}\sup_{\complement B_t}|x|^{-\beta+\alpha-n-\gamma}$$
$$=C_{18}\sup_{t>0}t^{\frac{n+\gamma}{q}-\lambda+\alpha-n-\gamma-\beta}<\infty\Longleftrightarrow\alpha-\beta-\lambda=n+\gamma-\frac{n+\gamma}{q}.$$

Now in the case $p=1$ the second part of Theorem A leads us to the inequality

$$J_3\leq\frac{C_{19}}{\tau}\int_{\mathbb{R}^n_+}|x|^\beta|f(x)|x_n^\gamma dx,$$

where the positive constant C_{19} is independent of f.

We now, we estimate J_2.

From $\beta+\lambda\geq0$ and Theorem 1 we get

$$J_2=\left(\sum_{k\in\mathbb{Z}}\int_{\{x\in D_k:\,|x|^{-\lambda}\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)|\,T^y|x|^{\alpha-n-\gamma}y_n^\gamma dy>\tau/3\}}x_n^\gamma dx\right)^{1/q}$$

$$\le\left(\sum_{k\in\mathbb{Z}}\int_{\{x\in D_k:\,\int_{B_{2|x|}\setminus B_{|x|/2}}|f(y)||y|^\beta\,T^y|x|^{\alpha-\beta-\lambda-n-\gamma}y_n^\gamma dy>c\tau\}}x_n^\gamma dx\right)^{1/q}$$

$$\le\left(\sum_{k\in\mathbb{Z}}\int_{\{x\in D_k:\,\left|I_{\alpha-\beta-\lambda,\gamma}\left(f(\cdot)|\cdot|^\beta\chi_{\widetilde{D_k}}\right)(x)\right|>c\tau\}}x_n^\gamma dx\right)^{1/q}$$

$$\le\left(\sum_{k\in\mathbb{Z}}\left(\frac{C_{20}}{\tau}\int_{\widetilde{D_k}}|f(x)||x|^\beta\,x_n^\gamma dx\right)^q\right)^{1/q}\le\left(\frac{C_{21}}{\tau}\int_{\mathbb{R}^n_+}|x|^\beta|f(x)|x_n^\gamma dx\right)^{1/q}.$$

■

From Theorems 2 and 3 we get the following

THEOREM 4. *Let* $0<\alpha<n+\gamma$, $1\le p\le q<\infty$, $\beta<\frac{n+\gamma}{p'}$ ($\beta\le 0$, *if* $p=1$), $\lambda<\frac{n+\gamma}{q}$ ($\lambda\le 0$, *if* $q=\infty$), $\alpha\ge\beta+\lambda\ge 0$ ($\beta+\lambda>0$, *if* $p=q$). *Then:*

1) If $1<p<\frac{n+\gamma}{\alpha-\beta-\lambda}$, *then condition* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha-\beta-\lambda}{n+\gamma}$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{p,|x|^\beta,\gamma}(\mathbb{R}^n_+)$ *to* $L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$.

2) If $p=1$, *then condition* $1-\frac{1}{q}=\frac{\alpha-\beta-\lambda}{n+\gamma}$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,|x|^\beta,\gamma}(\mathbb{R}^n_+)$ *to* $WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$.

P r o o f. *Sufficiency* of Theorem 4 follows from Theorems 2 and 3.

Necessity. 1) Suppose that the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,|x|^\beta,\gamma}(\mathbb{R}^n_+)$ to $L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$ and $1<p<\frac{n+\gamma}{\alpha-\beta-\lambda}$.

Define $f_t(x)=:f(tx)$ for $t>0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,|x|^\beta,\gamma}}=t^{-\frac{n+\gamma}{p}-\beta}\,\|f\|_{L_{p,|x|^\beta,\gamma}}\quad,\quad(I_{\alpha,\gamma}f_t)(x)=t^{-\alpha}I_{\alpha,\gamma}f(tx),$$

and

$$\|I_{\alpha,\gamma}f_t\|_{L_{q,|x|^{-\lambda},\gamma}}=t^{-\alpha-\frac{n+\gamma}{q}+\lambda}\,\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}}.$$

Since the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,|x|^\beta,\gamma}(\mathbb{R}^n_+)$ to $L_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$, we have

$$\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}}\le C\|f\|_{L_{p,|x|^\beta,\gamma}},$$

where C is independent of f . Then we get

$$\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} = t^{\alpha+\frac{n+\gamma}{q}-\lambda}\,\|I_{\alpha,\gamma}f_t\|_{L_{q,|x|^{-\lambda},\gamma}}$$
$$\leq Ct^{\alpha+\frac{n+\gamma}{q}-\lambda}\|f_t\|_{L_{p,|x|^{\beta},\gamma}} = Ct^{\alpha+\frac{n+\gamma}{q}-\lambda-\frac{n+\gamma}{p}-\beta}\|f\|_{L_{p,|x|^{\beta},\gamma}}.$$

If $\frac{1}{p}-\frac{1}{q} < \frac{\alpha-\beta-\lambda}{n+\gamma}$, then for all $f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+)$ we have $\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} = 0$ as $t \to 0$.

If $\frac{1}{p}-\frac{1}{q} > \frac{\alpha-\beta-\lambda}{n+\gamma}$, then for all $f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+)$ we have $\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} = 0$ as $t \to \infty$.

Therefore we get the equality $\frac{1}{p} - \frac{1}{q} = \frac{\alpha-\beta-\lambda}{n+\gamma}$.

2) Suppose that the operator $I_{\alpha,\gamma}$ is bounded from $L_{1,|x|^{\beta},\gamma}(\mathbb{R}^n_+)$ to $WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$. It is easy to show that

$$\|f_t\|_{L_{1,|x|^{\beta},\gamma}} = t^{-n-\gamma-\beta}\,\|f\|_{L_{1,|x|^{\beta},\gamma}} \quad , \quad (I_{\alpha,\gamma}f_t)(x) = t^{-\alpha}(I_{\alpha,\gamma}f)(tx),$$

and

$$\|I_{\alpha,\gamma}f_t\|_{WL_{q,|x|^{-\lambda},\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}+\lambda}\,\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}}.$$

By the boundedness of $I_{\alpha,\gamma}$ from $L_{1,|x|^{\beta},\gamma}(\mathbb{R}^n_+)$ to $WL_{q,|x|^{-\lambda},\gamma}(\mathbb{R}^n_+)$, we have

$$\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}} \leq C\|f\|_{L_{1,|x|^{\beta},\gamma}},$$

where C is independent of f . Then we have

$$(I_{\alpha,\gamma}f_t)_{*,\gamma}(\tau) = t^{-n-\gamma}(I_{\alpha,\gamma}f)_{*,\gamma}(t^{\alpha}\tau),$$

$$\|I_{\alpha,\gamma}f_t\|_{WL_{q,|x|^{-\lambda},\gamma}} = t^{-\alpha-\frac{n+\gamma}{q}+\lambda}\,\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}},$$

and

$$\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}} = t^{\alpha+\frac{n+\gamma}{q}-\lambda}\|I_{\alpha,\gamma}f_t\|_{WL_{q,|x|^{-\lambda},\gamma}}$$
$$\leq Ct^{\alpha+\frac{n+\gamma}{q}-\lambda}\|f_t\|_{L_{1,|x|^{\beta},\gamma}} = Ct^{\alpha+\frac{n+\gamma}{q}-\lambda-n-\gamma-\beta}\|f\|_{L_{1,|x|^{\beta},\gamma}}.$$

If $1-\frac{1}{q} < \frac{\alpha-\beta-\lambda}{n+\gamma}$, then for all $f \in L_{1,|x|^{\beta},\gamma}$ we have $\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}} = 0$ as $t \to 0$.

If $1-\frac{1}{q} > \frac{\alpha-\beta-\lambda}{n+\gamma}$, then for all $f \in L_{1,|x|^{\beta},\gamma}$ we have $\|I_{\alpha,\gamma}f\|_{WL_{q,|x|^{-\lambda},\gamma}} = 0$ as $t \to \infty$.

Therefore we get the equality $1 - \frac{1}{q} = \frac{\alpha-\beta-\lambda}{n+\gamma}$.

Thus Theorem 4 is proved. ■

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